Positive definiteness of fourth order three dimensional symmetric tensors

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Abstract

For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, we show the analytic sufficient and necessary conditions of its positive definiteness. By applying these conclusions, several strict inequalities is bulit for ternary quartic homogeneous polynomials.

Keywords: Positive definiteness, Fourth order tensors, Homogeneous polynomial.

1. Introduction

One of the most direct applications of positive definite tensors is to verify the vacuum stability of the Higgs scalar potential model [1, 2]. Qi [3] first used the consept of positive definiteness for a symmetric tensor when the order is even integer.

Definition 1.1. *Let $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ be an mth order n dimensional symmetric tensor. $\mathcal{T}$ is called*

(i) **positive semi-definite** *([3]) if m is an even number and in the Euclidean space $\mathbb{R}^n$, its associated Homogeneous polynomial*

$$\mathcal{T}x^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} t_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \geq 0;$$

(ii) **positive definite** *([3]) if m is an even number and $\mathcal{T}x^m > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.*

Clearly, a positive semi-definite tensor coincides with a positive semi-definite matrix if $m = 2$. It is well-known that Sylvester's Criterion can efficiently identify the positive (semi-)definiteness of a matrix. The positive definiteness of a 4th order 2 dimensional symmetric tensor, (or positivity condition of a quartic univariate polynomial) may trace back to ones of Refs. Rees [4] , Lazard [5] Gadem-Li [6], Ku [7] and Jury-Mansour [8]. Untill to 2005, Wang-Qi [9] improved their proof, and perfectly gave analytic necessary and sufficient conditions. However, the above

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result depends on the discriminant of such a quartic polynomial. Hasan-Hasa [10] claimed that a necessary and sufficient condition of positive definiteness was proved without the discriminant. However, there is a problem in their argumentations. In 1998, Fu [11] pointed out that Hasan-Hasan's results are sufficient only. Recently, Guo[12] showed a new necessary and sufficient condition without the discriminant. Very recently, Qi-Song-Zhang[13] gave a new necessary and sufficient condition other than the above results. For more detail about applications of these results, see Song-Qi [14] also.

In 2005, Qi [3] gave that the sign of all H-(Z-)eigenvalue of a even order symmetric tensor can verify the positive definitiveness of such a higher order tensor. Subsequently, Ni-Qi-Wang [15] provided a method of computing the smallest eigenvalue for checking positive definiteness of a 4th order 3 dimensional tensor. Ng-Qi-Zhou [16] presented an algorithm of the largest eigenvalue of a nonnegative tensor. For a 4th order 3 dimensional symmetric tensor, Song [17] proved several sufficient conditions of its positive definiteness. Until now, an analytic necessary and sufficient condition has not been found for positive (semi-)definiteness for a 4th order 3 dimensional symmetric tensor.

In this paper, we mainly dicuss analytic necessary and sufficient conditions of positive definiteness of a class of 4th order 3-dimensional symmetric tensors (Theorem 3.1). Furthermore, several strict inequalities of ternary quartic homogeneous polynomial (Corollary 3.2) are built.

2. Copositivity of 4th order 2-dimensional symmetric tensors

Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor. Then for $x = (x_1, x_2)^\top$,

$$\mathcal{T}x^4 = t_{1111}x_1^4 + 4t_{1112}x_1^3x_2 + 6t_{1122}x_1^2x_2^2 + 4t_{1222}x_1x_2^3 + t_{2222}x_2^4. \tag{2.1}$$

Let

$$\begin{aligned}\Delta =& 4\times 12^3(t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2)^3\\ &- 72^2\times 6^2(t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1111}t_{1222}^2 - t_{1112}^2t_{2222})^2\\ =& 4\times 12^3(I^3 - 27J^2),\end{aligned}$$

where

$$\begin{aligned}I =& t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2,\\ J =& t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1111}t_{1222}^2 - t_{1112}^2t_{2222}.\end{aligned}$$

and hence, the sign of Δ is the same as one of $(I^3 - 27J^2)$. Ulrich-Watson [18] presented the analytic conditions of the nonnegativity of a quartic and univariate polynomial in $\mathbb{R}_+$. Qi-Song-Zhang [13] also gave the nonnegativity and positivity of a quartic and univariate polynomial in $\mathbb{R}$, which means the positive (semi-)definitiveness of 4th order 2-dimensional tensor [2].

Lemma 2.1 ([2, 13]). *A 4th-order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ is positive definite*

if and only if

$$(\boldsymbol{I})\quad \begin{cases} I^3-27J^2=0,\ \ t_{1112}\sqrt{t_{2222}}=t_{1222}\sqrt{t_{1111}},\\ 2t_{1112}^2+t_{1111}\sqrt{t_{1111}t_{2222}}=3t_{1111}t_{1122}<3t_{1111}\sqrt{t_{1111}t_{2222}};\\ I^3-27J^2>0,\ \ |t_{1112}\sqrt{t_{2222}}-t_{1222}\sqrt{t_{1111}}|\le\sqrt{6t_{1111}t_{1122}t_{2222}+2\sqrt{(t_{1111}t_{2222})^3}},\\ (i)\ -\sqrt{v_{1111}v_{2222}}\le 3v_{1221}\le 3\sqrt{v_{1111}v_{2222}};\\ (ii)\ t_{1221}>\sqrt{t_{1111}t_{2222}}\ \ and\\ |t_{1112}\sqrt{t_{2222}}+t_{1222}\sqrt{t_{1111}}|\le\sqrt{6t_{1111}t_{1122}t_{2222}-2\sqrt{(t_{1111}t_{2222})^3}}. \end{cases}$$

A 4th-order 2-dimensional symmetric tensor $\mathcal{T}=(t_{ijkl})$ is positive semidefinite if and only if

$$(\boldsymbol{II})\quad \begin{cases} I^3-27J^2\ge 0,\ \ |t_{1112}\sqrt{t_{2222}}-t_{1222}\sqrt{t_{1111}}|\le\sqrt{6t_{1111}t_{1122}t_{2222}+2\sqrt{(t_{1111}t_{2222})^3}},\\ (i)\ -\sqrt{t_{1111}t_{2222}}\le 3t_{1122}\le 3\sqrt{t_{1111}t_{2222}};\\ (ii)\ t_{1122}>\sqrt{t_{1111}t_{2222}}\ and\\ |t_{1112}\sqrt{t_{2222}}+t_{1222}\sqrt{t_{1111}}|\le\sqrt{6t_{1111}t_{1122}t_{2222}-2\sqrt{(t_{1111}t_{2222})^3}}. \end{cases}$$

Lemma 2.2. *Let $\mathcal{T}=(t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $|t_{ijkl}|=1$ and $t_{1111}=t_{2222}=1$. Then*

(i) $\mathcal{T}$ is positive semidefinite if and only if $t_{1122}=1$;

(ii) $\mathcal{T}$ is positive definite if and only if $t_{1122}=1$ and $t_{1112}t_{1222}=-1$.

Proof. (i) It follows from Lemma 2.1 (II) that $\mathcal{T}$ is positive semidefinite if and only if

$$I^3-27J^2\ge 0,\ \ |t_{1112}-t_{1222}|\le\sqrt{6t_{1122}+2}\text{ and }-1\le 3t_{1122}\le 3.$$

Since $|t_{ijkl}|=1$, then which means $t_{1122}=1$ and either $t_{1112}t_{1222}=1$,

$$I^3-27J^2=(1-4+3)^3-27(1+2-1-1-1)^2=0,$$

$$|t_{1112}-t_{1222}|=0<\sqrt{6t_{1122}+2}=\sqrt{8};$$

or $t_{1112}t_{1222}=-1$,

$$I^3-27J^2=(1+4+3)^3-27(1-2-1-1-1)^2>0,$$

$$|t_{1112}-t_{1222}|=2<\sqrt{6t_{1122}+2}=\sqrt{8}.$$

So $\mathcal{T}$ is positive semidefinite if and only if $t_{1122}=1$.

(ii) It follows from Lemma 2.1 (I) that $\mathcal{T}$ is positive definite if and only if

$$I^3-27J^2=0,\ \ t_{1112}=t_{1222},\ \ 2t_{1112}^2+1=3t_{1122}<3;$$
$$I^3-27J^2>0,\ \ |t_{1112}-t_{1222}|\le\sqrt{6t_{1122}+2}\text{ and }-1\le 3t_{1122}\le 3.$$

Since $3=2t_{1112}^2+1=3t_{1122}<3$ can't hold, then $\mathcal{T}$ is positive definite if and only if $t_{1122}=1$ and $t_{1112}t_{1222}=-1$. This completes the proof.

3. Positive definiteness of 4th order 3-dimensional symmetric tensors

Theorem 3.1. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{T}$ is positive definite if and only if

$$\textbf{(III)} \qquad \begin{cases} t_{iijj} = 1 \text{ and } t_{ijjj}t_{iiij} = -1 \text{ for all } i, j \in \{1, 2, 3\}, i \neq j \text{ and either} \\ t_{1123} = t_{1223} = t_{1233} = 1, \text{ or} \\ \text{two of } \{t_{1123}, t_{1223}, t_{1233}\} \text{ are } -1. \end{cases}$$

Proof. Necessity. Without loss the generality, let $x_1 = 0$ for $x = (x_1, x_2, x_3)\top$. Now,

$$\mathcal{T}x^4 = x_2^4 + x_3^4 + 6t_{2233}x_2^2x_3^2 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3,$$

and then, it follows from Lemma 2.2 that the positive definiteness of $\mathcal{T}$ implies

$$t_{2233} = 1 \text{ and } t_{2223}t_{2333} = -1.$$

Therefore, the condition that $t_{iijj} = 1$ and $t_{ijjj}t_{iiij} = -1$ for all $i, j \in \{1, 2, 3\}, i \neq j$ is necessary.

Suppose the remainder of these conditions can't hold when $\mathcal{T}$ is positive definite, then there may be two cases.

Case 1. $t_{1123} = t_{1223} = t_{1233} = -1$. Let $x = (1, 1, 1)^\top$. Then we have

$$\begin{aligned} \mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 \\ &+ 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3 \\ &- 12x_1^2x_2x_3 - 12x_1x_2^2x_3 - 12x_1x_2x_3^2 \\ =& 21 + 4 \times 3 - 4 \times 3 - 12 - 12 - 12 = -15 < 0; \end{aligned}$$

Case 2. There is only one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$. We might as well take $t_{1123} = t_{1223} = 1$ and $t_{1233} = -1$. For $x = (1, 1, -1)^\top$, we have

$$\begin{aligned} \mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 \\ &+ 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3 \\ &+ 12x_1^2x_2x_3 + 12x_1x_2^2x_3 - 12x_1x_2x_3^2 \\ =& 21 + 4 \times 3 - 4 \times 3 - 12 - 12 - 12 = -15 < 0. \end{aligned}$$

This is a contradiction to the positive definiteness of $\mathcal{T}$, and hence, the remainder conditions are necessary also.

Sufficiency. Let $t_{1122} = t_{1133} = t_{2233} = t_{1222} = t_{2333} = t_{1113} = 1$ and $t_{1112} = t_{1333} = t_{2223} = -1$ without loss the generality. $\mathcal{T}x^4$ may be rewritten as follows,

$$\begin{aligned} \mathcal{T}x^4 =& (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3) \\ &+ 12(t_{1123} - 1)x_1^2x_2x_3 + 12(t_{1223} - 1)x_1x_2^2x_3 + 12(t_{1233} - 1)x_1x_2x_3^2 \\ =& (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2) \\ &+ 12(t_{1123} + 1)x_1^2x_2x_3 + 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} - 1)x_1x_2x_3^2 \\ =& (x_1 - x_2 + x_3)^4 + 8(x_1x_2^3 + x_2x_3^3 - x_1^3x_3) \\ &+ 12(t_{1123} + 1)x_1^2x_2x_3 + 12(t_{1223} - 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2 \\ =& (x_2 + x_3 - x_1)^4 + 8(x_1^3x_3 + x_1x_2^3 - x_2^3x_3) \\ &+ 12(t_{1123} - 1)x_1^2x_2x_3 + 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2. \end{aligned}$$

(1) Two of $\{t_{1123}, t_{1223}, t_{1233}\}$ equal to -1. We might take $t_{1123} = t_{1223} = -1$ and $t_{1233} = 1$. Let $y = (x_1^2, x_2^2, x_3^3, x_1x_2, x_1x_3, x_2x_3)^\top$. Then

$$\mathcal{T}x^4 = y^\top My,$$

here

$$M = \begin{pmatrix} 1 & 1 & 2 & -2 & 2 & -3 \\ 1 & 1 & 2 & -3 & -2 & -2 \\ 2 & 2 & 1 & -3 & -2 & 2 \\ -2 & -3 & -3 & 4 & -3 & 3 \\ 2 & -2 & -2 & -3 & 2 & 3 \\ -3 & -2 & 2 & 3 & 3 & 2 \end{pmatrix}.$$

It follows from the well-know Sylvester's Criterion that the matrix M is positive semi-definite, and hence,

$$\mathcal{T}x^4 = y^\top My \geq 0.$$

Now, we only show that $\mathcal{T}x^4 = 0$ implies $x = 0$. In fact, rewriting $\mathcal{T}x^4$ as follow,

$$\mathcal{T}x^4 = (x_1 + x_2 - x_3)^4 + 8(x_1^3x_3 + x_2x_3^3 - x_1^3x_2),$$

Then the solutions of the system $\mathcal{T}x^3 = \frac{1}{4}\nabla\mathcal{T}x^4 = 0$ is stationary points of the function $\mathcal{T}x^4$. That is,

$$\mathcal{T}x^3 = \frac{1}{4}\nabla\mathcal{T}x^4 = \begin{pmatrix} \sum\limits_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l \end{pmatrix} = \begin{pmatrix} (x_1 + x_2 - x_3)^3 + 2(3x_1^2x_3 - 3x_1^2x_2) \\ (x_1 + x_2 - x_3)^3 + 2(x_3^3 - x_1^3) \\ -(x_1 + x_2 - x_3)^3 + 2(x_1^3 + 3x_2x_3^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and hence,

$$(x_1 + x_2 - x_3)^3 = -2(3x_1^2x_3 - 3x_1^2x_2) = -2(x_3^3 - x_1^3) = 2(x_1^3 + 3x_2x_3^2).$$

Therefore, $x_1 = x_2 = x_3 = 0$, and so, the function $\mathcal{T}x^4$ has a unique stationary point $O(0, 0, 0)$, which is unique minimum point. So, $\mathcal{T}x^4 > 0$ for all $x \neq 0$. That is, $\mathcal{T}$ is positive definite.

(2) $t_{1123} = t_{1223} = t_{1233} = 1$. Similarly, for $y = (x_1^2, x_2^2, x_3^3, x_1x_2, x_1x_3, x_2x_3)^\top$, we also have

$$\mathcal{T}x^4 = y^\top Ay,$$

here

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 & 2 & 3 \\ 1 & 1 & 1 & 3 & -2 & -2 \\ 1 & 1 & 1 & 3 & -2 & 2 \\ -2 & 3 & 3 & 4 & 3 & 3 \\ 2 & -2 & -2 & 3 & 4 & 3 \\ 3 & -2 & 2 & 3 & 3 & 4 \end{pmatrix}.$$

By Sylvester's Criterion, the matrix M is positive semi-definite, and so,

$$\mathcal{T}x^4 = y^\top Ay \geq 0.$$

Rewriting $\mathcal{T}x^4$ as follow,

$$\mathcal{T}x^4 = (x_1 + x_2 + x_3)^4 - 8(x_1^3x_2 + x_1x_3^3 + x_2^3x_3).$$

Then the stationary points of the function $\mathcal{T}x^4$ is the solutions of the system,

$$\mathcal{T}x^3 = \frac{1}{4}\nabla\mathcal{T}x^4 = 0, \text{ i.e., } \begin{cases} (x_1 + x_2 + x_3)^3 - 2(3x_1^2x_2 + x_3^3) = 0, \\ (x_1 + x_2 + x_3)^3 - 2(3x_2^2x_3 + x_1^3) = 0, \\ (x_1 + x_2 + x_3)^3 - 2(3x_1x_3^2 + x_2^3) = 0. \end{cases}$$

and hence,

$$(x_1 + x_2 + x_3)^3 = 2(3x_1^2x_2 + x_3^3) = 2(3x_2^2x_3 + x_1^3) = 2(3x_1x_3^2 + x_2^3).$$

This yields $x_1 = x_2 = x_3 = 0$. Therefore, this unique stationary point $O(0, 0, 0)$ is is unique minimum point of $\mathcal{T}x^4$. That is, $\mathcal{T}x^4 = 0$ implies $x = 0$. So, $\mathcal{T}$ is positive definite. This completes the proof.

By applying Theorems 3.1, the following strict inequalities are established easily for ternary quartic homogeneous polynomials.

Corollary 3.2. *If* $(x_1, x_2, x_3) \neq (0, 0, 0)$*, then*

(i) $(x_1 + x_2 + x_3)^4 > 8(x_1x_3^3 + x_1^3x_2 + x_2^3x_3)$;

(ii) $(x_1 + x_2 - x_3)^4 > 8(x_1^3x_2 - x_1^3x_3 - x_2x_3^3)$;

(iii) $(x_1 - x_2 + x_3)^4 > 8(x_1^3x_3 - x_1x_2^3 - x_2x_3^3)$;

(iv) $(x_1 - x_2 - x_3)^4 > 8(x_2^3x_3 - x_1^3x_3 - x_1x_2^3)$.

Furthermore, these strict inequalities still hold if $x_1^3x_2$ *and* $x_1x_2^3$ *are exchangeable, or* $x_1^3x_3$ *and* $x_1x_3^3$ *are exchangeable, or* $x_2x_3^3$ *and* $x_2^3x_3$ *are exchangeable.*

4. Conclusions

For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, the analytic necessary and sufficient conditions are established for its positive definiteness. Several strict inequalities of ternary quartic homogeneous polynomial are built by means of these analytic conditions.

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